Splines on Wheel Graphs Over $\mathbb{Z}$ mod $p^k\mathbb{Z}$

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Abstract

Given a graph **G** whose edges are labelled by ideals of a commutative ring **R**, a generalized spline is a labelling of each vertex by a commutative ring element so that adjacent vertices differ by an element of the ideal associated to the edge.These generalized splines form a sub ring of a product of copies of **R**. So they form a module over **R,** termed as generalised spline modules.The module of generalized splines contain a free sub module whose rank is the number of vertices in **G**. We find a generating set of flow-up classes for wheel graphs over the ring $\mathbb{Z}/p^k\mathbb{Z}$, where p is prime.Also we classify splines on cycles and wheel graphs over the ring $\mathbb{Z}/m\mathbb{Z}$ when **m** has few prime factors and find a generating set of flow up classes on these graphs over $\mathbb{Z}/m\mathbb{Z}$.

I. Introduction

In this paper we extend the work done by Nealy Bowden and JuliannaTymoczko on cycles [1] to classify splines on wheel graphs, find a minimum generating set of flow-up classes over $\mathbb{Z}/p^k\mathbb{Z}$, where p is a prime and classify splines on cycles over $\mathbb{Z}/m\mathbb{Z}$ if **m** has few prime factors

In various areas of mathematics, a smooth curve is created by piecing together polynomials so that at the point where two polynomials meet their derivatives upto certain order are equal. Mathematically a spline is a collection of polynomials on the faces of a polyhedral complex that agree (modulo power of a linear form) on the intersection of two faces ([5], [6], [7], [8], [9]).

Mathematicians chose the term splines to refer to the piecewise polynomial functions used to create smooth curves. Splines also have a rich history in Homological and Commutative Algebra as well as Geometry and Topology ([5], [8], [10]).

An integer generalized spline is a set of vertex labels on an edge-labeled graph that satisfy the condition that if two vertices are joined by an edge, the vertex labels are congruent modulo the edge label [Def.2.1] (Refer [2]).

The ring $\mathbb{Z}/m\mathbb{Z}$ is a finite ring which is not an integral domain. Thus the generalised spline modules over $\mathbb{Z}/m\mathbb{Z}$ must have minimum generating sets -namely a generating set with smallest possible size.The structure theorem for finite abelian groups [4]shows that finite modules are generally not free, but the minimum generating sets function like bases except that each element **b** of the minimum generating set has a scalar c_b satisfying

$$c_b.b=0$$

Over **ℤ/mℤ** these minimum generating sets can be smaller than expected. Over a domain we know that the module of splines contain a free submodule of rank atleast the number of vertices [2], and over a principal ideal domain the module of splines is always free with rank the number of vertices.There are at most **n** elements in the minimum generating set for splines **mod m** on a graph with n vertices Theorem4.1, [1], The rank of the ℤ-module of splines is defined to be the number of elements of a minimum generating set.

2. Preliminaries

1. G: a graph, defined as a set of vertices V and edges E, assumed throughout to befinite with no multiple edgesbetween vertices.

2. R: a commutative ring with identity 1.

3. I: the set of ideals in R.

4. α: an edge–labelling function on G that assigns a nonzero element of I to each edgein E.

5. (G, α): an edge–labeled graph.

6. $\mathbf{R_G}$: the ring of generalized splines on (G,α).

7. p: a generalized spline, satisfying the edge condition over the graph **G**.

Definition 2.1 (**Edge condition)**: Let **G(V,E)** be a finite graph. Let **R** be a commutative ring with identity. Let **α: E** $\rightarrow$ **{ideals in R}** be a function that labels the edges of **G** with ideals in **R**. The splines on **G** are elements $\mathbf{f} \in \mathbf{R}^{|V|}$ such that for each edge **{uv}** ϵ **E**, we have: $\mathbf{f_u - f_v \in (\alpha\ (uv))}$
The collection of splines over the graph **G** with edge-labelling **α** is denoted $\mathbf{R_{G,\alpha}}$ or just $\mathbf{R_G}$ if the edge-labelling is clear. In this paper the base ring is the quotient ring R = **ℤ/mℤ**. Every ideal in **ℤ/mℤ** is principal so we typically describe an edge-label **(a)** by the generator **a** ϵ **ℤ/mℤ**.

Remark 2.2: We generally assume that the edges of our graphs are not labelled with 0 or with units. If the edge $\mathbf{e = v_1 v_2}$ is labeled with a unit, it does not restrict the splines on the graph since $\mathbf{v_1 \equiv v_2 \bmod 1}$ is always true. If an edge $\mathbf{e = v_1 v_2}$ is labelled zero it tells us that for every spline **p** the values
$\mathbf{p_{v1} = p_{v2}}$.

The notion of flow-up splines, which generalizes the concept of a triangular generating set from linear algebra, is defined as follows:

Definition 2.3(Flow-up splines): Given a graph **G** with an ordered set of vertices **V** $\mathbf{= (v_1, v_2, \ldots, v_n)}$ a flow-up spline for a vertex $\mathbf{v}_i$ is a spline $\mathbf{f^i}$ for which$\mathbf{f^i}_{\mathbf{vi}} \mathbf{= 0,}$ whenever k < i**.**

Typically the order is chosen consistently with a direction on the edges of the graph.

Definition 2.4 (constant flow up spline): A constant flow up spline in $\mathbb{Z}/m\mathbb{Z}$ is a flow-up spline **p** for which there exists an element $n_i \epsilon\ \mathbb{Z}/m\mathbb{Z}$ such that $p_{v_i}\ \epsilon\ \{0, n_i\}$, for each $v_i\ \epsilon\ V$.

The graph we discuss most in this manuscript is the wheelgraph with **n+1** vertices, which we label as shown in [Fig.1].

3. Splines over $\mathbb{Z}/p^k\mathbb{Z}$.

In this section we use the following proposition 2.4 from [2], corollary 3.14 from [1], which are as follows:

Proposition[2]: R_G is a ring with unit **1,** defined by 1_v=1 for each vertex $v\ \epsilon\ V$.

Corollary[1]:Let **m** be an integer. Let **G** be an edge-labelled graph and let **G*** be a graph obtained from **G** by adding a vertex **v** and some edges between **v** and vertices in **G**. Each spline on the expanded graph **G*** consists of the sum of a spline coming from **G** and a spline supported exactly on the new vertex.

Here we construct an edge labelled wheel graph W_{n+1} from a cycle graph C_n by adding one vertex v_{n+1} in the interior of the cycle C_n and corresponding **n** edges between v_{n+1} and **n** vertices of C_n. This new vertex is adjacent to all the vertices of C_n.

So in wheel graph the vertex labels $v_1, v_2, v_3, \ldots, v_n$ satisfy the edge conditions of C_n and the vertex labels $v_1, v_2, \ldots\ldots, v_{n+1}$ satisfy the edge conditions of $l_1, l_2, \ldots, l_n, l_{n+1}, l_{n+2}, \ldots, l_{2n}$[Def2.1],where l_i's are shown in Fig1.

The following theorems 5.1[1], 5.2[1] and corollary 5.3[1] are used for proving our **Theorem 3.1**

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Theorem[1]: Let **p** be a prime number. If **G** is an edge-labelled graph over $\mathbb{Z}/p\mathbb{Z}$, with no edges labelled zero then every vertex-labelling over $\mathbb{Z}/p\mathbb{Z}$ is a spline on **G**.

Theorem[1]: If **G** is a connected graph such that every edge of **G** is labelled with **(a),** where **a** is an element of the ring **R**, then a minimum generating set for R_G is

$$B(R_G) = \left\{ \begin{pmatrix} 1 \\ 1 \\ . \\ . \\ . \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ . \\ . \\ 0 \\ 0 \\ a \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ . \\ . \\ 0 \\ a \\ 0 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ . \\ . \\ a \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdots\cdots \begin{pmatrix} a \\ . \\ . \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Corollary[1]: Let **G** be a graph and **p** a prime number. Then splines on **G** over $\mathbb{Z}/p^2\mathbb{Z}$ are generated by the minimum generating set $B(R_G)$.

Using the above results and Theorem 5.4 and corollary 5.5[1], we have shown that there exists a minimum generating set for wheel graphs over $\mathbb{Z}/p^2\mathbb{Z}$ for an arbitrary **n**.

Extending these results on cycles to wheel graphs $\mathbf{W_{n+1}}$ whose edges are labelled by some powers of $\mathbf{a} \in \mathbf{R}$, we have

Theorem 3.1: Let **a** be a zero divisor in $\mathbb{Z}/m\mathbb{Z}$. Suppose all of the edges of $\mathbf{W_{n+1}}$ are labelled with powers of **a**. Without loss of generality assume that $\mathbf{a^{k1}}$ is the minimal power in the set and that $\mathbf{a^{k1}}$ is the label on the edges $\mathbf{l_n, l_{n+1}, l_{n+2}, \ldots, l_{2n}}$. So the set of edge labels is $\mathbf{(a^{k1}, a^{k2}, a^{k3}, \ldots, a^{k2n})}$. Then the following set generates all splines on $\mathbf{W_{n+1}}$.

$$B(R_{W_{n+1}}) = \left\{ \begin{pmatrix} 1\\1\\1\\.\\.\\.\\1\\1 \end{pmatrix} \begin{pmatrix} l_1\\l_1\\.\\.\\.\\.\\l_1\\0 \end{pmatrix} .. \begin{pmatrix} l_i\\l_i\\.\\.\\.\\l_i\\0\\0 \end{pmatrix} .. \begin{pmatrix} l_{n-1}\\l_{n-1}\\.\\.\\.\\0\\0\\0 \end{pmatrix} \ldots \begin{pmatrix} l_n\\.\\.\\.\\.\\0\\0\\0 \end{pmatrix} \right\}$$

Proof: We need to verify that every element in our generating set is a spline on $\mathbf{R_{Wn+1}}$ and that every possible spline on $\mathbf{R_{Wn+1}}$ can be written in terms of elements of our generating set.

The trivial spline is a spline by definition. Notice that every other element of **B** is of the form $\mathbf{(l_i, l_i, \ldots, l_i, 0, \ldots, 0)^T}$. The difference between any pair of adjacent vertices is **0** around every edge except around the edges $\mathbf{l_i}$ and $\mathbf{l_{2n}}$. The spline conditions are trivially satisfied for each pair of adjacent vertices that differ by **0**. The difference over the other two edges is $\mathbf{l_i}$. Notice that $\mathbf{l_i}$ divides itself and recall our convention that the $\mathbf{n^{th}}$, $\mathbf{(n+1)^{th}}, \ldots, \mathbf{(2n)^{th}}$ edges are labelled with $\mathbf{a^{k1}}$ which divides all other edge-labels by assumption. Also l_n divides all other edge labels. Thus the spline conditions are satisfied at every edge.

Theorem 5.4[1] shows that every element $\mathbf{f} \in \mathbf{R_{W_{n+1}}}$ can be written as a linear combination of the splines in **B**.

Corollary 2.11[1] shows that this set is minimum, proving the claim.

□

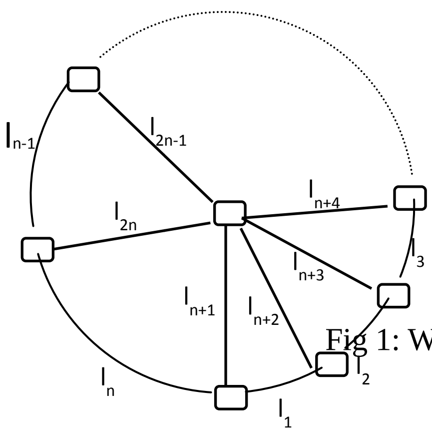

Fig 1: Wheel Graph

Corollary 3.2: Let $\mathbf{W_{n+1}}$ be the wheel graph on **n+1** vertices, let **p** be a prime number and let **k** be any positive integer. Then the splines on $\mathbf{R_{W_{n+1}}}$ over □/$\mathbf{p^k}$ □ are generated by the minimum generating set **B** in the above result.

Proof: The only possible edge labels over □/$\mathbf{p^k}$ □ are

$$\{(p), (p^2), (p^3), \ldots\ldots, (p^{k-1})\}$$

By rotating the edge-labelled graph we can assume that the edge $\mathbf{l_n}$ is labelled with the least power. This rotation induces an isomorphism on the ring of splines. Thus the above result gives a minimum generating set for $\mathbf{R_{W_{n+1}}}$, over □/$\mathbf{p^k}$□.

Example3.3 We give a set of constant flow up splines for a $\mathbf{W_5}$ over □/$\mathbf{2^5}$ □

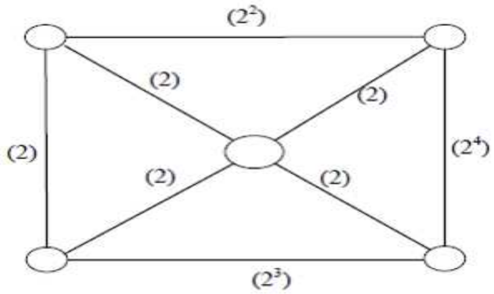

Fig 2: Wheel graph W_5

$$B(R(W_5)) = \left\{ \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} \begin{pmatrix}2^3\\2^3\\2^3\\2^3\\0\end{pmatrix} \begin{pmatrix}2^4\\2^4\\2^4\\0\\0\end{pmatrix} \begin{pmatrix}2^2\\2^2\\0\\0\\0\end{pmatrix} \begin{pmatrix}2\\0\\0\\0\\0\end{pmatrix} \right\}$$

4. Splines on cycles over □/m□

In this section,we construct flow-up bases for the generalised spline modules on cycles on C_n, over the ring □ /**m**□, for $\mathbf{m = m_1, m_2, \ldots\ldots, m_r}$,a prime factorization of **m**.

Theorem 4.1: Let $\mathbf{C_n}$ be a cycle with **n** vertices, and **R** be the ring □/**m**□, where $\mathbf{m = m_1m_2}$, $\mathbf{m_1}$ and $\mathbf{m_2}$ are primes. Let the edges $\mathbf{l_i}$ of $\mathbf{C_n}$ be labelled by either $\mathbf{m_1}$ or $\mathbf{m_2}$, such that both $\mathbf{m_1}$ and $\mathbf{m_2}$ appear as edge labels at least once. Then the following set **B** is a flow up generating set for the spline module $\mathbf{R_{C_n}}$.

$$B(R(C_n)) = \left\{ \begin{pmatrix}1\\1\\1\\.\\.\\.\\1\\1\end{pmatrix} \begin{pmatrix}l_{n-1}l_n\\l_{n-2}l_{n-1}\\.\\.\\.\\l_2l_3\\l_1l_2\\0\end{pmatrix} \begin{pmatrix}l_{n-1}l_n\\.\\.\\.\\l_3l_4\\l_2l_3\\0\\0\end{pmatrix}, \ldots, \begin{pmatrix}l_{n-1}l_n\\.\\0\\.\\.\\0\\0\\0\end{pmatrix} \right\}$$

Proof: First we note that each element in the set B satisfies the edge conditions over the cycle $\mathbf{C_n}$, and hence is a spline. Next, we want to show that every arbitrary spline **f** in $\mathbf{R_{C_n}}$ can be expressed as a linear combination of elements in **B**. We use the method of induction for this, over the number of leading zeroes in **f**. If **f** has no leading zero, then $\mathbf{f - f_{v_1}(1, 1, \ldots, 1)}$ is a spline with one leading zero. Suppose, **f** has **i** leading zeroes. Then the restriction of **f** over the vertex $\mathbf{v_{i+1}}$, i.e., $\mathbf{f_{v_{i+1}}}$ is a multiple of $\mathbf{l_i l_{i+1}}$. Let $\mathbf{c_i} = \mathbf{f_{v_{i+1}}/l_i l_{i+1}}$. Then, $\mathbf{f - c_i(0, 0, \ldots, 0, l_i l_{i+1}, l_{i+1} l_{i+2}, \ldots, l_{n-1} l_n)}$ is a spline with **i+1** leading zeroes.

Thus the set **B** forms a generating set for $\mathbf{R_{C_n}}$. □

In the above case, i.e, when **m** has only two prime factors the generating set **B** may not be minimum. It will lose a rank whenever two adjacent edges of $\mathbf{C_n}$ are labelled with distinct primes $\mathbf{m_1}$ and $\mathbf{m_2}$.

However, if **m** has three or more prime factors and the edges of $\mathbf{C_n}$ are labelled such that each prime factor of **m** appears at least once in the edge labelling of $\mathbf{C_n}$, then the above set **B** will be minimum.

Example 4.2: We give a set of flow- up splines for $\mathbf{C_5}$ over ℤ**/(2x3x5)** ℤ, which forms a generating set for $\mathbf{R_{C5}}$.

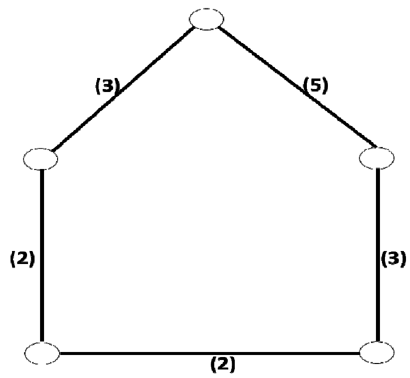

Fig 3: Cycle with 5 vertices

$$B(R(C_5)) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 3\times 2 \\ 5\times 3 \\ 3\times 5 \\ 2\times 3 \\ 0 \end{pmatrix} \begin{pmatrix} 3\times 2 \\ 5\times 3 \\ 3\times 5 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 3\times 2 \\ 5\times 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 3\times 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Remark4.3:We observe that when **m** has more than **2** prime factors, the generating set doesn't loose a rank since every vertex in cycles has only two edges.

5. Splines on wheel graphs over ℤ/mℤ

In this section we extend the method used in the previous

section to construct a minimum generating set for the wheel graph. As noted earlier, the wheel graph $\mathbf{W_{n+1}}$ is obtained from the cycle $\mathbf{C_n}$ by adding a vertex $\mathbf{v_{n+1}}$ to the set of vertices
$\{\mathbf{v_1, v_2, \ldots, v_n}\}$ and the edges $\{\mathbf{l_{n+1}, l_{n+2}, \ldots, l_{2n}}\}$ to the set of edges $\{\mathbf{l_1, l_2, \ldots, l_n}\}$ [Fig.1]. Then we have the following theorem:

Theorem 5.1: Let $\mathbf{W_{n+1}}$ be a wheel graph with **n+1** vertices and consider the quotient ring **Z/mZ**, where $\mathbf{m = m_1 m_2}$, where $\mathbf{m_1}$ and $\mathbf{m_2}$ are primes. Label the edges of $\mathbf{W_{n+1}}$ in such a way that $\mathbf{m_1}$ and $\mathbf{m_2}$ appear at least once as edge labelling. Then the following set **B** is a generating set of the spline module $\mathbf{R_{W_{n+1}}}$, over the base ring **Z/mZ**.

$$B(R_{(W_{n+1})}) = \left\{ \begin{pmatrix} 1\\1\\1\\ \cdot\\ \cdot\\ \cdot\\ 1\\ 1 \end{pmatrix} \begin{pmatrix} l_{n+1}l_{n+2}..l_{2n}\\ l_{n-1}l_n l_{2n}\\ \cdot\\ \cdot\\ \cdot\\ l_2 l_3 l_{n+3}\\ l_1 l_2 l_{n+2}\\ 0 \end{pmatrix} \begin{pmatrix} l_{n+1}l_{n+2}\cdots l_{2n}\\ l_{n-1}l_n l_{2n}\\ \cdot\\ \cdot\\ l_3 l_4 l_{n+4}\\ l_2 l_3 l_{n+3}\\ 0\\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} l_{n+1}l_{n+2}\cdots l_{2n}\\ l_{n-1}l_n l_{2n}\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ 0\\ 0 \end{pmatrix} \begin{pmatrix} l_{n+1}l_{n+2}\cdots l_{2n}\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ 0\\ 0 \end{pmatrix} \right\}$$

Proof: Since the wheel graph is obtained by adding a vertex and **n** edges to the cycle graph $\mathbf{C_n}$, we want to show that the edge conditions are satisfied for the edges $\mathbf{l_{n+1}, l_{n+2}, \ldots, l_{2n}}$, by each spline in **B**.

Since the last vertex $\mathbf{v_{n+1}}$ is labelled with the element $\mathbf{l_{n+1}l_{n+2}\ldots l_{2n}}$, which is a product of the labelling on the edges which where added to $\mathbf{C_n}$ to get the wheel graph $\mathbf{W_{n+1}}$, the difference of the vertex label on $\mathbf{v_{n+1}}$with any vertex $\mathbf{v_i}$ will be a multiple of $\mathbf{l_{n+i}}$. This immediately proves our claim that the edge conditions are satisfied.

Also, the above set generates any arbitrary spline **f** in $\mathbf{R_{W_{n+1}}}$, can be easily proved by inducting over the number of leading zeroes in the elements of **B**. □

We observe that when $\mathbf{m = m_1 m_2}$, where $\mathbf{m_1, m_2}$ are primes, all splines on wheel graphs are trivial splines, whenever $\mathbf{m_1}$ and $\mathbf{m_2}$ both appear at least once as edge labels for the edges adjacent at each of it's vertices.

Example 5.2: We give an example of wheel graph with 6 vertices over □ **/ (2 x 3)**□

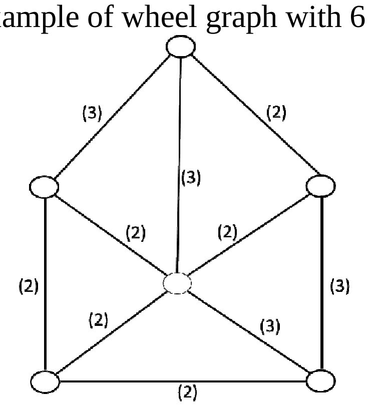

Fig 4: Wheel graph with 6 vertices

$$B(R_{W_6}) = \left\{ \begin{pmatrix} 1\\1\\1\\1\\1\\1 \end{pmatrix} \begin{pmatrix} 2\times3\times2\times3\times2\\ 3\times2\times2\\ 3\times3\times2\\ 2\times2\times3\\ 2\times3\times3\\ 0 \end{pmatrix} \begin{pmatrix} 2\times3\times2\times3\times2\\ 3\times2\times2\\ 3\times3\times2\\ 2\times2\times3\\ 0\\ 0 \end{pmatrix} \begin{pmatrix} 2\times3\times2\times3\times2\\ 3\times2\times2\\ 3\times3\times2\\ 0\\ 0\\ 0 \end{pmatrix} \begin{pmatrix} 2\times3\times2\times3\times2\\ 3\times2\times2\\ 0\\ 0\\ 0\\ 0 \end{pmatrix} \begin{pmatrix} 2\times3\times2\times3\times2\\ 0\\ 0\\ 0\\ 0\\ 0 \end{pmatrix} \right\}$$

In this example, if we take the vertex labels on each vertex **mod (2□3)**, we see that each spline over the wheel graph will be a trivial spline.

However, if at some vertex of the above graph, if all edges meeting at a point are either labelled as only $\mathbf{m_1}$ or only $\mathbf{m_2}$, then the generating set will have non trivial elements and hence the splines over the wheel graph will also be non trivial.

Using similar algorithm, we can construct a generating set over a wheel graph, when **m** has more number of prime factors. Here we completely characterise the situation, when the generating set of the ring $\mathbf{R_{W_{n+1}}}$, will be minimum.

Theorem 5.3: Let $\mathbf{W_{n+1}}$ be a wheel graph, with vertices $\mathbf{v_1, v_2, ..., v_{2n}}$ and edges $\mathbf{l_1, l_2, ..., l_{2n}}$ as in [Fig.1], and $\mathbf{m = m_1 m_2 ... m_r}$, where $\mathbf{m_1, m_2, ..., m_r}$ are primes, Let each edge of the wheel graph be labelled by the prime factors of **m**. Then, we can get the generating set **B** of flow up splines by taking the product of the edge lables meeting at a vertex, as the vertex lable of the corresponding vertex as in Theorem 5.1. The above set will be minimum whenever the number of prime factors of **m** is greater than **n**, i.e, the number of vertices in the cycle graph $\mathbf{C_n}$.

Proof: The proof follows from the fact that exactly three edges meet at a vertex lying in the cycle graph $\mathbf{C_n}$, and the interior vertex $\mathbf{v_{n+1}}$ is adjacent to exactly **n** edges in the wheel graph $\mathbf{W_{n+1}}$. □

Here we give some examples of wheel graphs when **m =2□3□5, m= 2□3□5 □7** and **m = 2□3□5□7□11** over □/m□.

Example 5.4: Wheel graph with 5 vertices when **m = 2□3□5**

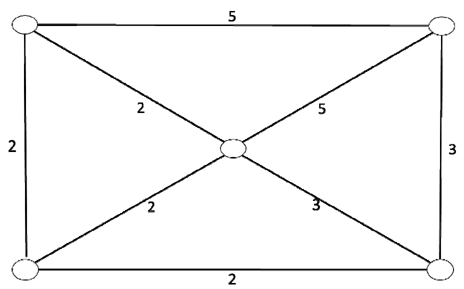

Fig 5: Wheel graph with 5 vertices

$$B(R_{(W_5)}) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2\times3\times5\times2 \\ 5\times2\times2 \\ 3\times5\times5 \\ 2\times3\times3 \\ 0 \end{pmatrix} \begin{pmatrix} 2\times3\times5\times2 \\ 5\times2\times2 \\ 3\times5\times5 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2\times3\times5\times2 \\ 5\times2\times2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2\times3\times5\times2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Generating set looses a rank when **m= 2□3□5**

Example 5.5: Wheel graph with 6 vertices **m = 2□3□5**

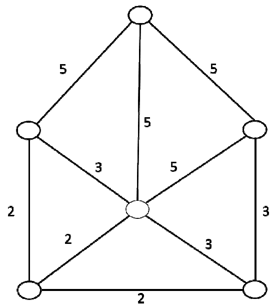

Fig 5: Wheel graph with 6 vertices

$$B(R_{(W_6)}) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2\times3\times5\times5\times3 \\ 5\times3\times2 \\ 5\times5\times5 \\ 3\times5\times5 \\ 2\times3\times3 \\ 0 \end{pmatrix} \begin{pmatrix} 2\times3\times5\times5\times3 \\ 5\times3\times2 \\ 5\times5\times5 \\ 3\times5\times5 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2\times3\times5\times5\times3 \\ 5\times3\times2 \\ 5\times5\times5 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2\times3\times5\times5\times3 \\ 5\times3\times2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2\times3\times5\times5\times3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Generating set looses a rank when **m = 2 □ 3 □ 5**

Example 5.6: Wheel graph with 5 vertices when**m= 2 □ 3 □ 5 □ 7**

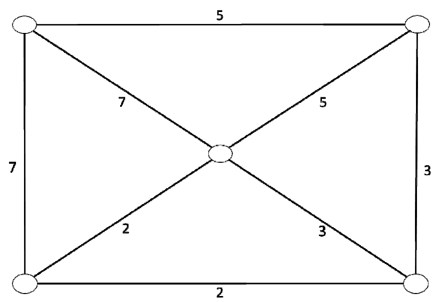

Fig 6: wheel graph with 5 vertices

$$B(R_{(W_5)}) = \left\{ \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} \begin{pmatrix}2\times3\times5\times7\\5\times7\times2\\3\times5\times5\\2\times3\times5\\0\end{pmatrix} \begin{pmatrix}2\times3\times5\times7\\5\times7\times2\\3\times5\times5\\0\\0\end{pmatrix} \begin{pmatrix}2\times3\times5\times7\\5\times7\times2\\0\\0\\0\end{pmatrix} \begin{pmatrix}2\times3\times5\times7\\0\\0\\0\\0\end{pmatrix} \right\}$$

Generating set looses a rank when **m = 2 □ 3 □ 5 □ 7**

Example 5.7: Here we give a Wheel graph with 5 vertices for **m = 2□ 3□ 5 □ 7 □ 11**

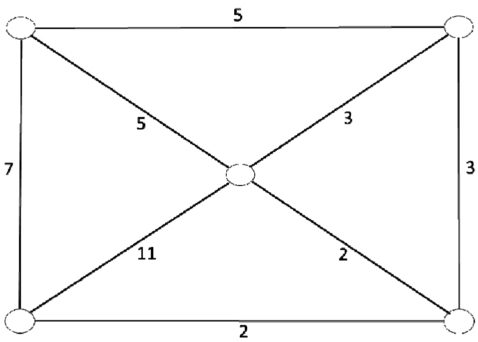

Fig 7: Wheel graph with 5 vertices

$$B(R_{(W_5)}) = \left\{ \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} \begin{pmatrix}11\times2\times3\times5\\5\times5\times7\\3\times3\times5\\2\times2\times3\\0\end{pmatrix} \begin{pmatrix}11\times2\times3\times5\\5\times5\times7\\3\times3\times5\\0\\0\end{pmatrix} \begin{pmatrix}11\times2\times3\times5\\5\times5\times7\\0\\0\\0\end{pmatrix} \begin{pmatrix}11\times2\times3\times5\\0\\0\\0\\0\end{pmatrix} \right\}$$

Generating set does not loose arank when **m = 2 □ 3 □ 5 □ 7 □ 11**

Remark 5.7: In wheel graphs when we exclude the central vertex, there are ***n-1*** vertices and we observe that if the number of prime factors in **m** are greater than ***n-1***, the splines on wheel graphs do not loose any rank over □ **/m□.**

6. Conclusions

We conclude our work with finding an algorithm for writing the generating set which acts as a basis for the generalised spline modules for cycle graphs, taking the base ring as the quotient ring of integers modulo **m,** whenever $\mathbf{m = m_1 m_2 ... m_r}$, where each $\mathbf{m_i}$ is a prime. The method is extendable to a generating set for the wheel graphs which is viewed as a graph extention to the cycle graph. Also, we noted that when the number of prime factors of **m** exceeds the number of vertices in the underlying cycle graph, the generating set is minimum. However, it may loose rank whenever **m** has fewer prime factors depending upon the labelling of the edges. This method is very systematic over the existing methods used in [1], and hence leads to a number of open questions as to whether it can be extended to other families of graphs as well as for base ring modulo powers of primes.

7. References

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