# Adjacent Vertex Sum Polynomial on Perfect Factographs

**A. M. Anto, P. Paul Hawkins**
Assistant Professor of Mathematics, Malankara Catholic College, India.
Assistant Professor of Mathematics, VTM College of Arts and Science, India.

---

## Abstract

The Adjacent Vertex Sum Polynomial of the graph is defined as $S(G,x) = \sum_{i=o}^{\Delta(G)} n_{\Delta(G)-i} x^{\alpha_{\Delta(G)-i}}$, where $n_{\Delta(G)-i}$ is the sum of the number of adjacent vertices of all the vertices of degree $\Delta(G) - i$ and $\alpha_{\Delta(G)-i}$ is the sum of the degree of adjacent vertices of all the vertices of degree $\Delta(G) - i$. In this paper I found the Adjacent Vertex Sum Polynomial of degree splitting and Total degree splitting graph of Perfect Factograph and Integral Perfect Factograph.

**2010 AMS Classification: 05C12.**

**KEYWORDS:** Factograph, Integral Perfect Factograph, Degree splitting, Total degree splitting.

## Introduction:

Here I consider simple graphs only. The terms not defined here refer Frank Harary [7]. Using unique factorization for integers, every positive integer z can be written in the form $z = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, where $p_1, p_2, \dots, p_r$ are distinct primes and $\alpha_1, \alpha_2, \dots \alpha_r$ are positive integers. We can construct a graph G which is associated with this z. Positive divisors of z being a vertex set V, then the two distinct vertices of V are adjacent in G if their product is in V and the corresponding graph is called Factograph [6]. The graph obtained when $r = 1$, is called Perfect Factograph. In this Factograph taking both positive and negative divisors as vertex set we obtain the new class of graphs called Integral Factograph and the case $r = 1$ named as Integral Perfect Factograph. Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup \dots \cup S_i \cup T$, where each $S_i$ is a set of vertices having at least two vertices and having the same degree and $T = V \backslash \cup S_i$. The degree splitting graph of G denoted by $DS(G)$ and is obtained from G by adding the vertices $w_1, w_2, \dots, w_t$ and joining $w_i$ to each vertex of $S_i, 1 \le i \le t$. Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup \dots \cup S_i$, where each $S_i$ is a set of vertices having same degree. The total degree splitting graph of G denoted by $TDS(G)$ and is obtained from G by adding the vertices $w_1, w_2, \dots, w_t$ and joining $w_i$ to each vertex of $S_i, 1 \le i \le t$.

**Theorem 1.1:** Let $G = {p_1}^{\alpha_1}$ be the Perfect Factograph. The adjacent vertex sum polynomial of $DS(G)$ is given by $S(DS(G), x)$

$$= \begin{cases} \alpha_1 x^{(\alpha_1-1)+(\alpha_1-2)+\cdots+\left(\frac{\alpha_1}{2}+1\right)+\left(\frac{\alpha_1}{2}+1\right)+\cdots+1} + (\alpha_1-1)x^{\alpha_1+(\alpha_1-2)+\cdots+\left(\frac{\alpha_1}{2}+1\right)+\left(\frac{\alpha_1}{2}+1\right)+\cdots+2} \\ \quad + \cdots + 2\left(\frac{\alpha_1}{2}+1\right)x^{2\left[\alpha_1+(\alpha_1-1)+\cdots+\left(\frac{\alpha_1}{2}+1\right)+2\right]} + \cdots + x^{\alpha_1} + 2x^{\alpha_1+2}, \\ \quad \text{when } \alpha_1 \text{ is even.} \\ \alpha_1 x^{(\alpha_1-1)+(\alpha_1-2)+\cdots+\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)+\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)+\cdots+1} + (\alpha_1-1)x^{\alpha_1+(\alpha_1-2)+\cdots+\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)+\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)+\cdots+2} \\ \quad + \cdots + 2\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)x^{2\left[\alpha_1+(\alpha_1-1)+\cdots+\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)+2\right]} + \cdots + x^{\alpha_1} + 2x^{2\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)}, \\ \quad \text{when } \alpha_1 \text{ is odd.} \end{cases}$$

**Proof:**

**Case (i):** $\alpha_1$ is even.

Let G be a Perfect Factograph. $V = \{p_1^0, p_1^1, p_1^2, \ldots, p_1^{\frac{\alpha_1}{2}}, p_1^{\frac{\alpha_1}{2}+1}, \ldots, p_1^{\alpha_1}\}$ be the vertex set. we have the degrees G is $d(p_1^0) = \alpha_1, d(p_1^1) = \alpha_1 - 1, \ldots, d\left(p_1^{\frac{\alpha_1}{2}}\right) = \frac{\alpha_1}{2}, d\left(p_1^{\frac{\alpha_1}{2}+1}\right) = \frac{\alpha_1}{2}, \ldots, d(p_1^{\alpha_1}) = 1$. Note that all the vertices except $p_1^{\frac{\alpha_1}{2}}$ and $p_1^{\frac{\alpha_1}{2}+1}$ have different degrees. Therefore, we introduce a new vertex w and make the vertex wadjacent to the vertices $p_1^{\frac{\alpha_1}{2}}$ and $p_1^{\frac{\alpha_1}{2}+1}$. Consider the vertex $p_1^0$, here the sum of the adjacent vertices is nothing but the degree of $p_1^0$, that is $\alpha_1$ and the sum of the degree of adjacent vertices of $\alpha_1$ is $(\alpha_1-1)+(\alpha_1-2)+\cdots+\left(\frac{\alpha_1}{2}+1\right)+\left(\frac{\alpha_1}{2}+1\right)+\cdots+1$. Therefore, the term of the polynomial corresponding to the degree $\alpha_1$ is $\alpha_1 x^{(\alpha_1-1)+(\alpha_1-2)+\cdots+\left(\frac{\alpha_1}{2}+1\right)+\left(\frac{\alpha_1}{2}+1\right)+\cdots+1}$. Continuing like this, the vertices $p_1^{\frac{\alpha_1}{2}}$ and $p_1^{\frac{\alpha_1}{2}+1}$ have degree $\frac{\alpha_1}{2}+1$, the sum of the adjacent vertices of degree $\frac{\alpha_1}{2}+1$ is $2(\frac{\alpha_1}{2}+1)$ and sum of the degree of the adjacent vertices of degree $\frac{\alpha_1}{2}+1$ is $2\left[\alpha_1 + \right.$ α1−1+...+α12+1+2. We get the corresponding term for degree α12+1 is $2\left(\frac{\alpha_1}{2}+1\right)x^{2\left[\alpha_1+(\alpha_1-1)+\cdots+\left(\frac{\alpha_1}{2}+1\right)+2\right]}$. The last vertex $p_1^{\alpha_1}$ has degree 1, the sum of the adjacent vertices of degree 1 is 1 and sum of the degree of the adjacent vertices of degree 1 is $\alpha_1$. Therefore, we get the term $x^{\alpha_1}$. Finally, the new vertex w has degree 2 and the adjacent vertices have degree sum $\alpha_1 + 2$, which gives the term $2x^{\alpha_1+2}$. Now add all the terms corresponding to all the vertices of DS(G) gives that

$$S(G,x) = \alpha_1 x^{(\alpha_1-1)+(\alpha_1-2)+\cdots+\left(\frac{\alpha_1}{2}+1\right)+\left(\frac{\alpha_1}{2}+1\right)+\cdots+1}$$

$$+(\alpha_1-1)x^{\alpha_1+(\alpha_1-2)+\cdots+\left(\frac{\alpha_1}{2}+1\right)+\left(\frac{\alpha_1}{2}+1\right)+\cdots+2} + \cdots$$

$$+2\left(\frac{\alpha_1}{2}+1\right)x^{2\left[\alpha_1+(\alpha_1-1)+\cdots+\left(\frac{\alpha_1}{2}+1\right)+2\right]} + \cdots + x^{\alpha_1} + 2x^{\alpha_1+2}.$$

**Case (ii):** $\alpha_1$ is odd.

Let $G$ be a Perfect Factograph and $V=\left\{p_1^0, p_1^1, p_1^2, \ldots, p_1^{\left\lfloor\frac{\alpha_1}{2}\right\rfloor}, p_1^{\left\lceil\frac{\alpha_1}{2}\right\rceil}, \ldots, p_1^{\alpha_1}\right\}$ be a vertex set. we have the degrees of $G$ is $d(p_1^0)=\alpha_1$, $d(p_1^1)=\alpha_1-1, \ldots, d\left(p_1^{\left\lfloor\frac{\alpha_1}{2}\right\rfloor}\right)=\left\lceil\frac{\alpha_1}{2}\right\rceil, d\left(p_1^{\left\lceil\frac{\alpha_1}{2}\right\rceil}\right)=\left\lceil\frac{\alpha_1}{2}\right\rceil, \ldots, d(p_1^{\alpha_1})=1$. Note that all the vertices except $p_1^{\left\lfloor\frac{\alpha_1}{2}\right\rfloor}$ and $p_1^{\left\lceil\frac{\alpha_1}{2}\right\rceil}$ have different degrees. Therefore, we introduce a new vertex $w$ and make the vertex $w$ adjacent to the vertices $p_1^{\left\lfloor\frac{\alpha_1}{2}\right\rfloor}$ and $p_1^{\left\lceil\frac{\alpha_1}{2}\right\rceil}$. Consider the vertex $p_1^0$, here the sum of the adjacent vertices is nothing but the degree of $p_1^0$, that is $\alpha_1$ and the sum of the degree of adjacent vertices of $\alpha_1$ is $(\alpha_1-1)+(\alpha_1-2)+\cdots+\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)+\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)+\cdots+1$. Therefore, the term of the polynomial corresponding to the degree $\alpha_1$ is $\alpha_1 x^{(\alpha_1-1)+(\alpha_1-2)+\cdots+(\lceil\frac{\alpha_1}{2}\rceil+1)+(\lceil\frac{\alpha_1}{2}\rceil+1)+\cdots+1}$. Continuing like this, the vertices $p_1^{\left\lfloor\frac{\alpha_1}{2}\right\rfloor}$ and $p_1^{\left\lceil\frac{\alpha_1}{2}\right\rceil}$ have degree $\left\lceil\frac{\alpha_1}{2}\right\rceil+1$, the sum of the adjacent vertices of degree $\left\lceil\frac{\alpha_1}{2}\right\rceil+1$ is $2\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)$ and sum of the degree of the adjacent vertices of degree $\left\lceil\frac{\alpha_1}{2}\right\rceil+1$ is $2\left[\alpha_1+(\alpha_1-1)+\cdots+\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)+2\right]$. We get the corresponding term for degree $\left\lceil\frac{\alpha_1}{2}\right\rceil+1$ is $2\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)x^{2[\alpha_1+(\alpha_1-1)+\cdots+(\lceil\frac{\alpha_1}{2}\rceil+1)+2]}$. The last vertex $p_1^{\alpha_1}$ has degree 1, the sum of the adjacent vertices of degree 1 is 1 and sum of the degree of the adjacent vertices of degree 1 is $\alpha_1$. Therefore, we get the term $x^{\alpha_1}$.Finally, the new vertex $w$ has degree 2 and the adjacent vertices have degree sum $2\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)$, which gives the term $2x^{2(\lceil\frac{\alpha_1}{2}\rceil+1)}$. Now add all the terms corresponding to all the vertices of $G$ gives that $S(G,x)=\alpha_1 x^{(\alpha_1-1)+(\alpha_1-2)+\cdots+(\lceil\frac{\alpha_1}{2}\rceil+1)+(\lceil\frac{\alpha_1}{2}\rceil+1)+\cdots+1}$

$$+(\alpha_1-1)x^{\alpha_1+(\alpha_1-2)+\cdots+(\lceil\frac{\alpha_1}{2}\rceil+1)+(\lceil\frac{\alpha_1}{2}\rceil+1)+\cdots+2}+\cdots$$
$$+2\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)x^{2[\alpha_1+(\alpha_1-1)+\cdots+(\lceil\frac{\alpha_1}{2}\rceil+1)+2]}+\cdots+x^{\alpha_1}+2x^{2(\lceil\frac{\alpha_1}{2}\rceil+1)}.$$

**Example 1.2:** Consider the Perfect Factograph $p_1^6$, then the graph $DS(p_1^6)$ is depicted as follows;

$DS(p_1^6)$:

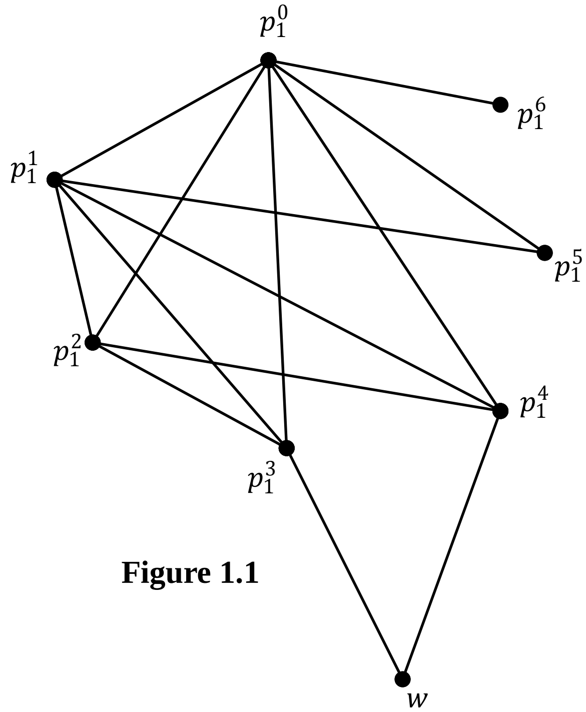


**Figure 1.1**

Here, $S(DS(p_1^6),x) = 6x^{20} + 5x^{20} + 4x^{19} + 4x^{17} + 4x^{17} + 2x^{11} + 2x^8 + x^6.$

$$= 11x^{20} + 4x^{19} + 8x^{17} + 2x^{11} + 2x^8 + x^6.$$

**Theorem 1.3:** Let $G = {p_1}^{\alpha_1}$ be an Integral Perfect Factograph. The adjacent vertex sum polynomial of $DS(G)$ is given by $S(DS(G),x)$

$$= \begin{cases} 2(2\alpha_1+2)x^{[(2\alpha_1+2)+2(2\alpha_1)+\cdots+2(\alpha_1+2)+2(\alpha_1+1)+\cdots+2(3)+2]} \\ +2(2\alpha_1)x^{[2(2\alpha_1+2)+(2\alpha_1)+\cdots+2(\alpha_1+2)+2(\alpha_1+1)+\cdots+2(5)+2]} + \cdots \\ +2(\alpha_1+2)x^{[2(2\alpha_1+2)+2(2\alpha_1)+\cdots+(\alpha_1+2)+2]} + \cdots + 2(3)x^{2(2\alpha_1+2)+2} \\ +2(x^{2(2\alpha_1+2)} + x^{2(2\alpha_1)} + \cdots + x^{2(\alpha_1+2)} + \cdots + x^{2(3)}), \\ \text{when } \alpha_1 \text{ is even.} \\ 2(2\alpha_1+2)x^{\left[(2\alpha_1+2)+2(2\alpha_1)+\cdots+2\left(2\left\lceil\frac{\alpha_1}{2}\right\rceil+2\right)+2\left(2\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)+\cdots+2(3)+2\right]} \\ +2(2\alpha_1)x^{\left[2(2\alpha_1+2)+(2\alpha_1)+\cdots+2\left(2\left\lceil\frac{\alpha_1}{2}\right\rceil+2\right)+2\left(2\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)+\cdots+2(5)+2\right]} + \cdots \\ +2\left(2\left\lceil\frac{\alpha_1}{2}\right\rceil+2\right)x^{\left[2(2\alpha_1+2)+2(2\alpha_1)+\cdots+\left(2\left\lceil\frac{\alpha_1}{2}\right\rceil+2\right)+2\right]} + \cdots \\ +2(3)x^{2(2\alpha_1+2)+2} + 2(x^{2(2\alpha_1+2)} + x^{2(2\alpha_1)} + \cdots \\ +x^{2\left(2\left\lceil\frac{\alpha_1}{2}\right\rceil+2\right)} + \cdots + x^{2(3)}) \\ \text{when } \alpha_1 \text{ is odd.} \end{cases}$$

**Proof:**

**Case (i):** $\alpha_1$ is even.

Let $G = {p_1}^{\alpha_1}$ be an Integral Perfect Factograph. Let $V = \{p_1^0, -p_1^0, p_1^1, -p_1^1, p_1^2, -p_1^2, \cdots, p_1^{\frac{\alpha_1}{2}}, -p_1^{\frac{\alpha_1}{2}}, p_1^{\frac{\alpha_1}{2}+1}, -p_1^{\frac{\alpha_1}{2}+1}, \dots, {p_1}^{\alpha_1}, -{p_1}^{\alpha_1}\}$ be the vertex set of G and order of V is $2(\alpha_1+1)$. we have the degrees of $G, d(p_1^0) = 2\alpha_1 + 1, d(-p_1^0) = 2\alpha_1 + 1, \dots, d\left(p_1^{\frac{\alpha_1}{2}}\right) = \alpha_1 + 1,\ d\left(-p_1^{\frac{\alpha_1}{2}}\right) = \alpha_1 + 1, d\left(p_1^{\frac{\alpha_1}{2}+1}\right) = \alpha_1, d\left(-p_1^{\frac{\alpha_1}{2}+1}\right) = \alpha_1, \dots,$

$d(p_1^{\alpha_1}) = 2, d(-p_1^{\alpha_1}) = 2$. Now we split the vertex set V as two different vertex sets such that $A = \{p_1^0,\ p_1^1,\ p_1^2, \cdots, p_1^{\frac{\alpha_1}{2}}, p_1^{\frac{\alpha_1}{2}+1}, \dots, {p_1}^{\alpha_1}\}$ and
$B = \left\{-p_1^0,\ -p_1^1, -p_1^2, \cdots, -p_1^{\frac{\alpha_1}{2}}, -p_1^{\frac{\alpha_1}{2}+1}, \dots, -{p_1}^{\alpha_1}\right\}$. Now, both A and B have order $\alpha_1 + 1$ and also that A and B have same degree sequence with distinct degrees. Therefore, we take $\alpha_1 + 1$ new vertices $\{w_0, w_1, w_2, \dots, w_{\alpha_1}\}$ and make adjacent each vertex $w_i$ to each vertex $p_1^i$ from A and $-p_1^i$ from B where $0 \le i \le \alpha_1$. Consequence of this, we get the degree splitting graph of Integral Perfect Factograph. Here, each new vertex $w_i$, where $0 \le i \le \alpha_1$ has degree 2 and each degree of the existing vertices of G increased by one, which gives the result.

**Case (ii):** $\alpha_1$ is odd.

Let $G = {p_1}^{\alpha_1}$be the Integral Perfect Factograph. Let $V = \{p_1^0, -p_1^0, p_1^1, -p_1^1, p_1^2, -p_1^2, \cdots, p_1^{\lfloor \frac{\alpha_1}{2} \rfloor}, -p_1^{\lfloor \frac{\alpha_1}{2} \rfloor}, p_1^{\lceil \frac{\alpha_1}{2} \rceil}, -p_1^{\lceil \frac{\alpha_1}{2} \rceil}, \dots, {p_1}^{\alpha_1}, -{p_1}^{\alpha_1}\}$ be the vertex set of G. By theorem 3.12, we have the degrees of G,

$$d(p_1^0) = 2\alpha_1 + 1, d(-p_1^0) = 2\alpha_1 - 1, \dots, d\left(p_1^{\lfloor \frac{\alpha_1}{2} \rfloor}\right) = \alpha_1 + 1,$$

$$d\left(-p_1^{\lfloor \frac{\alpha_1}{2} \rfloor}\right) = \alpha_1 + 1, d\left(p_1^{\lceil \frac{\alpha_1}{2} \rceil}\right) = \alpha_1, d\left(-p_1^{\lceil \frac{\alpha_1}{2} \rceil}\right) = \alpha_1, \dots,$$

$d(p_1^{\alpha_1}) = 2, d(-p_1^{\alpha_1}) = 2$. Now, we split the vertex set V as two different vertex sets such that

$A = \{p_1^0,\ p_1^1,\ p_1^2, \cdots, p_1^{\lfloor \frac{\alpha_1}{2} \rfloor}, p_1^{\lceil \frac{\alpha_1}{2} \rceil}, \dots, {p_1}^{\alpha_1}\}$ and
$B = \left\{-p_1^0,\ -p_1^1, -p_1^2, \cdots, -p_1^{\lfloor \frac{\alpha_1}{2} \rfloor}, -p_1^{\lceil \frac{\alpha_1}{2} \rceil}, \dots, -{p_1}^{\alpha_1}\right\}$. Now, both A and B have order $\alpha_1 + 1$ and also that A and B have same degree sequence with distinct degrees. Therefore, we take $\alpha_1 + 1$ new vertices $\{w_0, w_1, w_2, \dots, w_{\alpha_1}\}$ and make adjacent each vertex $w_i$ to each vertex $p_1^i$from A and $-p_1^i$from B where $0 \leq i \leq \alpha_1$. Consequence of this, we get the degree splitting graph of Integral Perfect Factograph. Here, each new vertex $w_i$ , where $0 \leq i \leq \alpha_1$ has degree 2 and each degree of the existing vertices of G increased by one, which gives the result.

**Example 1.4:** Consider the Integral Perfect Factograph $p_1^3$, then the graph $DS(p_1^3)$ is depicted as follows;

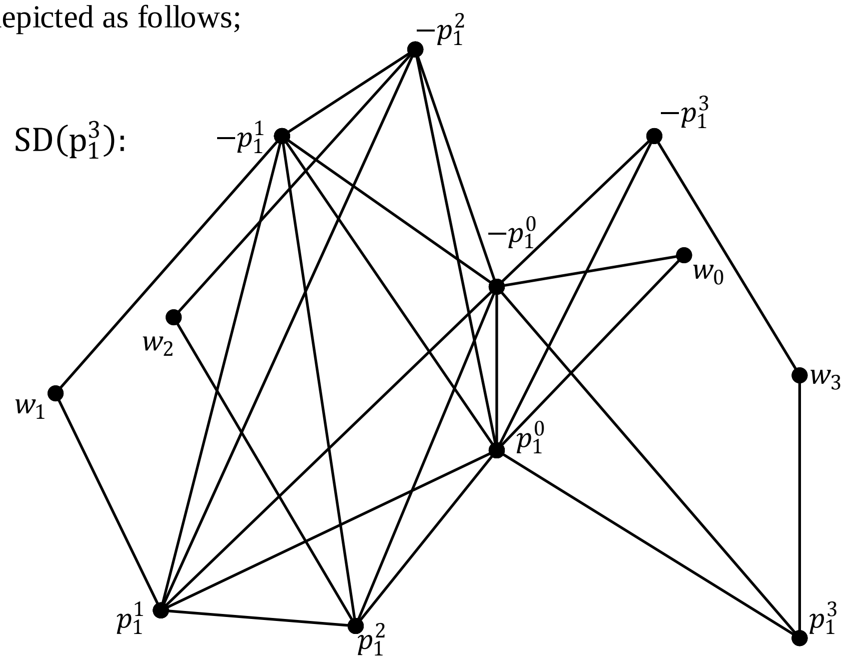


**Figure 1.2**

Now, $S(DS(G), x) = 8x^{38} + 8x^{38} + 6x^{34} + 6x^{34} + 5x^{30} + 5x^{30} + 3x^{18}$
$$+ 3x^{18} + 2x^{18} + 2x^{12} + 2x^{10} + 2x^6.$$

$$S(DS(G), x) = 16x^{38} + 12x^{34} + 10x^{30} + 6x^{18} + 2x^{18}\ +2x^{12} + 2x^{10} + 2x^6.$$

**Theorem 1.5:** Let $G = {p_1}^{\alpha_1}$ be the Perfect Factograph. The adjacent vertex sum polynomial of $TDS(G)$ is given by $S(TDS(G), x)$

$$= \begin{cases} (\alpha_1+1)x^{\alpha_1+(\alpha_1-1)+\cdots+\left(\frac{\alpha_1}{2}+1\right)+\left(\frac{\alpha_1}{2}+1\right)+\cdots+2+1} \\ +\alpha_1 x^{(\alpha_1+1)+\alpha_1+\cdots+\left(\frac{\alpha_1}{2}+1\right)+\left(\frac{\alpha_1}{2}+1\right)+\cdots+3+1} + \cdots \\ +2\left(\frac{\alpha_1}{2}+1\right)x^{\left[(\alpha_1+1)+\alpha_1+\cdots+\left(\frac{\alpha_1}{2}+1\right)+2\right]} + \cdots \\ +x^{(\alpha_1+1)+1} + x^{\alpha_1+1} + x^{\alpha_1} + \cdots + 2x^{\alpha_1+2} + \cdots + x, \\ \text{when } \alpha_1 \text{ is even.} \\ (\alpha_1+1)x^{\alpha_1+(\alpha_1-1)+\cdots+\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)+\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)+\cdots+2+1} \\ +\alpha_1 x^{(\alpha_1+1)+\alpha_1+\cdots+\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)+\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)+\cdots+3+1} + \cdots \\ +2\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)x^{\left[(\alpha_1+1)+\alpha_1+\cdots+\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)+2\right]} + \cdots \\ +x^{(\alpha_1+1)+1} + x^{\alpha_1+1} + x^{\alpha_1} + \cdots + 2x^{2\left(\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)} + \cdots + x, \\ \text{when } \alpha_1 \text{ is odd.} \end{cases}$$

**Proof:**

**Case (i):** $\alpha_1$ is even.

Let $G$ be a Perfect Factograph, $V = \{p_1^0, p_1^1, p_1^2, \ldots, p_1^{\frac{\alpha_1}{2}}, p_1^{\frac{\alpha_1}{2}+1}, \ldots, {p_1}^{\alpha_1}\}$ be the vertex set. we have the degrees of $G$ is $d(p_1^0) = \alpha_1, d(p_1^1) = \alpha_1 - 1, \ldots, d\left(p_1^{\frac{\alpha_1}{2}}\right) = \frac{\alpha_1}{2}, d\left(p_1^{\frac{\alpha_1}{2}+1}\right) = \frac{\alpha_1}{2}, \ldots, d(p_1^{\alpha_1}) = 1$. Note that all the vertices except $p_1^{\frac{\alpha_1}{2}}$ and $p_1^{\frac{\alpha_1}{2}+1}$ have different degrees. From the definition of total degree splitting graph, we introduce the $\alpha_1$ new vertices, among this $\alpha_1$ vertices one vertex is adjacent to the both vertices $p_1^{\frac{\alpha_1}{2}}$ and $p_1^{\frac{\alpha_1}{2}+1}$ and remaining $\alpha_1 - 1$ vertices adjacent to all the $\alpha_1 - 1$ vertices of $G$ except $p_1^{\frac{\alpha_1}{2}}$ and $p_1^{\frac{\alpha_1}{2}+1}$. In the resultant graph, each degree of existing vertices of $G$ has been increased by one, among new $\alpha_1$ vertices, each $\alpha_1 - 1$ vertices have degree one and one vertex has degree 2. This gives the adjacent vertex sum polynomial

$$S(TDS(G), x) = (\alpha_1+1)x^{\alpha_1+(\alpha_1-1)+\cdots+\left(\frac{\alpha_1}{2}+1\right)+\left(\frac{\alpha_1}{2}+1\right)+\cdots+2+1}$$
$$+\alpha_1 x^{(\alpha_1+1)+\alpha_1+\cdots+\left(\frac{\alpha_1}{2}+1\right)+\left(\frac{\alpha_1}{2}+1\right)+\cdots+3+1} + \cdots$$
$$+2\left(\frac{\alpha_1}{2}+1\right)x^{\left[(\alpha_1+1)+\alpha_1+\cdots+\left(\frac{\alpha_1}{2}+1\right)+2\right]} + \cdots + x^{(\alpha_1+1)+1}$$
$$+x^{\alpha_1+1} + x^{\alpha_1} + \cdots + 2x^{\alpha_1+2} + \cdots + x.$$

**Case (ii):** $\alpha_1$ is odd.

Let $G$ be a Perfect Factograph and $V = \{p_1^0, p_1^1, p_1^2, \ldots, p_1^{\left\lfloor\frac{\alpha_1}{2}\right\rfloor}, p_1^{\left\lceil\frac{\alpha_1}{2}\right\rceil}, \ldots, {p_1}^{\alpha_1}\}$ be a vertex set. we have the degrees of

G is $d(p_1^0)=\alpha_1, d(p_1^1)=\alpha_1-1,\dots,d\left(p_1^{\lfloor\frac{\alpha_1}{2}\rfloor}\right)=\lceil\frac{\alpha_1}{2}\rceil, d\left(p_1^{\lceil\frac{\alpha_1}{2}\rceil}\right)=\lceil\frac{\alpha_1}{2}\rceil,\dots,d(p_1^{\alpha_1})=1$. Note that all the vertices except $p_1^{\lfloor\frac{\alpha_1}{2}\rfloor}$ and $p_1^{\lceil\frac{\alpha_1}{2}\rceil}$ have different degrees. From the definition of total degree splitting graph, we introduce the $\alpha_1$ new vertices, among this $\alpha_1$ vertices one vertex is adjacent to the both vertices $p_1^{\lfloor\frac{\alpha_1}{2}\rfloor}$ and $p_1^{\lceil\frac{\alpha_1}{2}\rceil}$ and remaining $\alpha_1-1$ vertices adjacent to all the $\alpha_1-1$ vertices of G except $p_1^{\lfloor\frac{\alpha_1}{2}\rfloor}$ and $p_1^{\lceil\frac{\alpha_1}{2}\rceil}$. In the resultant graph, each degree of existing vertices of G has been increased by one, among new $\alpha_1$ vertices, each $\alpha_1-1$ vertices have degree one and one vertex has degree 2. This gives the adjacent vertex sum polynomial $S(TDS(G),x)=(\alpha_1+1)x^{\alpha_1+(\alpha_1-1)+\cdots+(\lceil\frac{\alpha_1}{2}\rceil+1)+(\lceil\frac{\alpha_1}{2}\rceil+1)+\cdots+2+1}$

$$+\alpha_1 x^{(\alpha_1+1)+\alpha_1+\cdots+(\lceil\frac{\alpha_1}{2}\rceil+1)+(\lceil\frac{\alpha_1}{2}\rceil+1)+\cdots+3+1}+\cdots$$

$$+2\left(\lceil\tfrac{\alpha_1}{2}\rceil+1\right)x^{[(\alpha_1+1)+\alpha_1+\cdots+(\lceil\frac{\alpha_1}{2}\rceil+1)+2]}+\cdots+x^{(\alpha_1+1)+1}$$

$$+x^{\alpha_1+1}+x^{\alpha_1}+\cdots+2x^{2(\lceil\frac{\alpha_1}{2}\rceil+1)}+\cdots+x.$$

**Example 1.6:** Consider the Perfect Factograph $p_1^5$, then the graph $TDS(p_1^5)$ is depicted as follows;

$TDS(p_1^5)$:

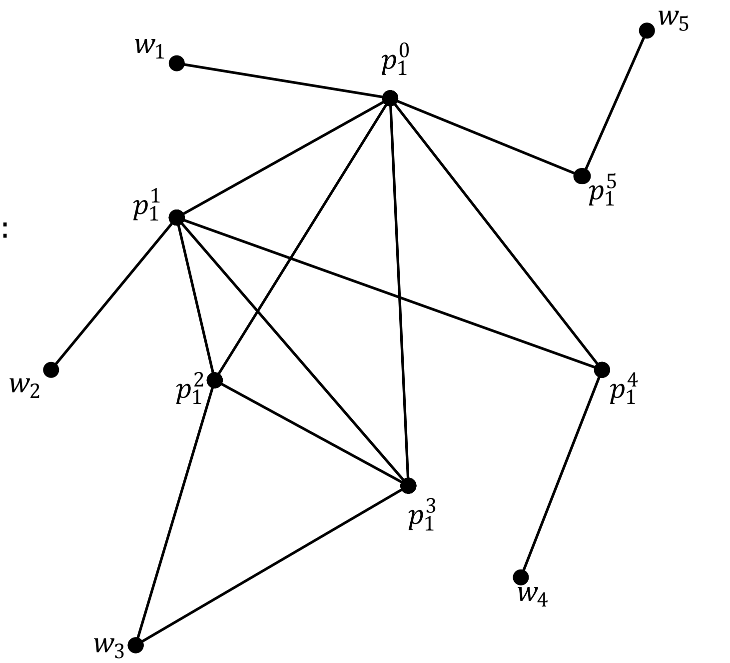


**Figure 1.3**

Here, $S(TDS(p_1^5),x)=6x^{19}+5x^{18}+4x^{17}+4x^{17}+3x^{12}+2x^7$

$$+x^6+x^5+2x^8+x^3+x^2.$$

$$S(TDS(p_1^5),x)=6x^{19}+5x^{18}+8x^{17}+3x^{12}+2x^7$$

$$+x^6+x^5+2x^8+x^3+x^2.$$

**Theorem 1.7:** Let $G = {p_1}^{\alpha_1}$ be an Integral Perfect Factograph. Then the adjacent sum vertex sum polynomial of $TDS(G)$ is given by

$$S(TDS(G),x) = \begin{cases} 2(2\alpha_1+2)x^{[(2\alpha_1+2)+2(2\alpha_1)+\cdots+2(\alpha_1+2)+2(\alpha_1+1)+\cdots+2(3)+2]} \\ +2(2\alpha_1)x^{[2(2\alpha_1+2)+(2\alpha_1)+\cdots+2(\alpha_1+2)+2(\alpha_1+1)+\cdots+2(5)+2]}+\cdots \\ +2(\alpha_1+2)x^{[2(2\alpha_1+2)+2(2\alpha_1)+\cdots+(\alpha_1+2)+2]}+\cdots+2(3)x^{2(2\alpha_1+2)+2} \\ +2(x^{2(2\alpha_1+2)}+x^{2(2\alpha_1)}+\cdots+x^{2(\alpha_1+2)}+\cdots+x^{2(3)}), \\ \text{when } \alpha_1 \text{ is even.} \\ 2(2\alpha_1+2)x^{\left[(2\alpha_1+2)+2(2\alpha_1)+\cdots+2\left(2\left\lceil\frac{\alpha_1}{2}\right\rceil+2\right)+2\left(2\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)+\cdots+2(3)+2\right]} \\ +2(2\alpha_1)x^{\left[2(2\alpha_1+2)+(2\alpha_1)+\cdots+2\left(2\left\lceil\frac{\alpha_1}{2}\right\rceil+2\right)+2\left(2\left\lceil\frac{\alpha_1}{2}\right\rceil+1\right)+\cdots+2(5)+2\right]}+\cdots \\ +2\left(2\left\lceil\frac{\alpha_1}{2}\right\rceil+2\right)x^{\left[2(2\alpha_1+2)+2(2\alpha_1)+\cdots+\left(2\left\lceil\frac{\alpha_1}{2}\right\rceil+2\right)+2\right]}+\cdots \\ +2(3)x^{2(2\alpha_1+2)+2}+2(x^{2(2\alpha_1+2)}+x^{2(2\alpha_1)}+\cdots \\ +x^{2\left(2\left\lceil\frac{\alpha_1}{2}\right\rceil+2\right)}+\cdots+x^{2(3)}) \\ \text{when } \alpha_1 \text{ is odd.} \end{cases}$$

**Proof:**

Let $G = {p_1}^{\alpha_1}$ be an Integral Perfect Factograph. From the definition of Degree splitting graph and Total degree splitting graph, we have $DS(G) \cong TDS(G)$. That is isomorphic graphs have same vertex polynomial. Therefore, $S(DS(G),x) = S(TDS(G),x)$.

**Result 1.8:** If $G = {p_1}^{\alpha_1}$ is an Integral Perfect Factograph, then

$S(DS(G),x) = S(TDS(G),x)$.

**References:**


[1] David. M. Burton "Elementary Number Theory", McGraw Hill Education.

[2] Douglas B. West “Introduction to Graph Theory” Second Edition, PHI Learning Private Limited, New Delhi.

[3] E. Ebin Raja Merly, A.M.Anto, “Adjacent Vertex Sum Polynomial of Factographs”, JGRMA- volume 2, No.11, November 2015.

[4] E. Ebin Raja Merly, E.Giftin Vedha Merly, A.M.Anto, “Degree Sequence and Clique number of Bi- Factograph and Tri-Factograph”, IJMTT-Volume 21- May 2015, pp 28-32.

[5] E. Ebin Raja Merly, E.Giftin Vedha Merly, A.M.Anto, “On Integral Perfect Factograph and Integral Bi-Factograph” IJMA-6(8), 2015, 47-50.

[6] E. Giftin Vedha Merly and N. Gnanadhas," On Factograph", International Journal of Mathematics Research, Volume 4, Number 2(2012), Page No. 125-131.

[7] Frank Harary, ”Graph Theory”, Narosa Publishing House, New Delhi, 2001.

[8] J.Devaraj, E.Sukumaran “On Vertex Polynomial”, International Journal of Math.sci&Engg Appls (IJMESA) Vol. 6 No. 1 (January, 2012), Page No. 371-380.

[9] S. S. Sandhya, C. Jeyasekaran, C. D. Raj (2013), “Harmonic Mean Labeling Of Degree Splitting Graphs” Bulletin of Pure and Applied Sciences, 32E, 99-112.